Hume's 'of Miracle' and Bayes' Theorem

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Bayes' Theorem

Partly as a response to Hume's famous skepticism about induction, Reverend Thomas Bayes (with help from his friend Richard Price) penned an 'Essay Towards Solving a Problem in the Doctrine of Chances' (1763), which articulated the following:

Note: Bayes' Theorem is a *theorem*: it follows from the axioms of probability and the ratio formula for conditional probabilities. As an exercise, try to prove it yourself!

BAYES' THEOREM. For any $X, E \in \mathcal{L}$, where $c(E) > 0$,

$$c(X \mid E) = \frac{c(E \mid X) \cdot c(X)}{c(E)} \tag{1}$$

Given *The Law of Total Probability*, the theorem can be rewritten as follows:

$$c(X \mid E) = \frac{c(E \mid X) \cdot c(X)}{c(E \mid X) \cdot c(X) + c(E \mid \neg X) \cdot c(\neg X)} \tag{2}$$

And, where $X, Y_1, Y_2, \ldots, Y_n \in \mathcal{L}$ form a partition,

$$c(X \mid E) = \frac{c(E \mid X) \cdot c(X)}{c(E \mid X) \cdot c(X) + c(E \mid Y_1) \cdot c(Y_1) + \cdots + c(E \mid Y_n) \cdot c(Y_n)} \tag{3}$$

$c(X \mid E)$ is called your *posterior*—it's your confidence in the hypothesis X *after* the evidence E has been supposed.

$c(E \mid X)$ is the *likelihood* of evidence E according to X.

$c(X)$ and $c(E)$ are your *priors*—they are your unconditional credences in a hypothesis and evidence before anything is supposed.

Version (2) and (3) are especially useful because they allow us to calculate the confirmatory support that evidence supplies for a hypothesis X by making use of information we might very well have available to us—e.g., the *likelihood* of that evidence according to each hypothesis, plus your *priors* in those hypotheses.

Hume's Argument Against Miracles

According to Hume, a miracle is (by definition) a violation of the laws of nature. But we have incredibly strong evidence—based on a tremendous amount of experience—for these laws.

Here's what he says:

> A miracle is a violation of the laws of nature; and as a firm and unalterable experience has established these laws, the proof against a miracle, from the very nature of the fact, is as entire as any argument from experience can possibly be imagined.

Hume isn't saying that experience always trumps eyewitness testimony; instead, he articulates the following maxim:

> HUME'S MAXIM: That no testimony is sufficient to establish a miracle, unless the testimony be of such a kind, that its falsehood would be more miraculous than the fact which it endeavors to establish.

Is there testimony of such a kind? Hume is skeptical:

> The many instances of forged miracles, and prophecies, and supernatural events, which, in all ages, have either been detected by contrary evidence, or which detect themselves by their absurdity, prove sufficiently the strong propensity of mankind to the extraordinary and the marvellous, and ought reasonably to beget a suspicion against all relations of this kind.

In brief (and to put it probabilistically), Hume thinks: you should assign very low credence to there having been a miracle; and eyewitness testimony just isn't reliable enough to raise your confidence very much.

Price's Response: the newspaper analogy

Richard Price responded to Hume as follows:

> The improbability of drawing a lottery in any particular assigned manner, independently of the evidence of testimony, or of our own sense, acquainting us that it *has* been drawn in that manner is such as exceeds all conception. And yet the testimony of a newspaper, or of any common man, is sufficient to put us out of doubt about it.

Roughly, if Hume is right, then we should never become confident about which lottery number won—after all, the chance of that particular number being drawn is incredibly low, and our evidence that it was the drawn number is the fallible testimony of the newspaper. But of course we can become confident in such things!

This is a *parity of reasoning argument.*

Arguments of this kind can be answered by finding a point of disanalogy between the two cases.

Dawid and Gillies' Bayesian Rebuttal

We have two scenarios. How confident should you be in miracles based on eyewitness testimony? How confident should you be that lottery ticket #n was drawn given that the newspaper reported it?

Dawid and Gillies argue that the two cases are not analogous.

$c(M \mid \text{“}M\text{”}) = ?$

$c(\#n \mid \text{“}\#n\text{”}) = ?$

The Lottery Ticket

Suppose there are 100,000 lottery tickets. The newspaper reports that ticket #n is the winner. Upon reading the newspaper, it's rational to become very confident that ticket #n is in fact the winner—even though, antecedently, it's very unlikely that #n would win.

Assuming that each ticket was equally likely to win, there was only a .01% of #n winning.

Using Bayes' Theorem, let's calculate your posterior.

#n	...	Ticket #n won.
“#n”	...	The newspaper reports that ticket #n won.

$$c(\#n \mid \text{“}\#n\text{”}) = \frac{c(\text{“}\#n\text{”} \mid \#n) \cdot c(\#n)}{c(\text{“}\#n\text{”})}$$

Let's suppose that the newspaper is very reliable: if a particular number wins, the paper is 99% likely to report so correctly.

For any ticket #x, $c(\text{“}\#x\text{”} \mid \#x) = .99$.

Let's also suppose that, on the off chance that the newspaper prints the incorrect number, it isn't more likely to print any one of the incorrect numbers than any other.

For all $\#y, \#z \neq \#x$, $c(\text{“}\#y\text{”} \mid \#x) = c(\text{“}\#z\text{”} \mid \#x)$.

Because the newspaper is 99% reliable, there is only a .01 chance that it reports an incorrect number. And because there are 100,000 tickets and only one can win, 99,999 are incorrect. So, for any $\#y \neq \#x$, $c(\text{“}\#y\text{”} \mid \#x) = \frac{.01}{99999}$.

Finally, let's suppose that each ticket was equally likely to win as any other.

Because there are 100,000 tickets, for each ticket $\#x$, $c(\#x) = \frac{1}{100{,}000} = .00001$.

From *The Law of Total Probability* (and because at most one ticket can win),

$$c(\text{“}\#n\text{”}) = c(\text{“}\#n\text{”} \mid \#n) \cdot c(\#n) + c(\text{“}\#n\text{”} \mid \#1) \cdot c(\#1) + c(\text{“}\#n\text{”} \mid \#2) \cdot c(\#2) + \ldots c(\text{“}\#n\text{”} \mid \#100,000) \cdot c(\#100,000)$$

This allows us to calculate your posterior using Bayes' Theorem.

$$c(\#n \mid \text{“}\#n\text{”}) = \frac{c(\text{“}\#n\text{”} \mid \#n) \cdot c(\#n)}{c(\text{“}\#n\text{”} \mid \#n) \cdot c(\#n) + c(\text{“}\#n\text{”} \mid \#1) \cdot c(\#1) + \ldots c(\text{“}\#n\text{”} \mid \#100,000) \cdot c(\#100,000))}$$

$$= \frac{.99 \cdot \frac{1}{10{,}000}}{c(\text{“}\#n\text{”} \mid \#n) \cdot c(\#n) + c(\text{“}\#n\text{”} \mid \#1) \cdot c(\#1) + \ldots c(\text{“}\#n\text{”} \mid \#100,000) \cdot c(\#100,000)}$$

$$= \frac{.99 \cdot \frac{1}{100{,}000}}{.99 \cdot \frac{1}{100{,}000} + \frac{.01}{99{,}999} \cdot \frac{1}{100{,}000} + \cdots + \frac{.01}{99{,}999} \cdot \frac{1}{100{,}000}}$$

Setting aside ticket $\#n$, there are 99,999 remaining tickets. So, there are 99,999 terms of $(\frac{.01}{99{,}999} \cdot \frac{1}{100{,}000})$.

$$= \frac{.99 \cdot \frac{1}{100{,}000}}{.99 \cdot \frac{1}{100{,}000} + 99,999 \cdot (\frac{.01}{99{,}999} \cdot \frac{1}{100{,}000})}$$

$$= \frac{.99 \cdot \frac{1}{100{,}000}}{.99 \cdot \frac{1}{100{,}000} + (.01 \cdot \frac{1}{100{,}000})}$$

$$= \frac{.99}{.99 + .01} = .99$$

Miracles

Suppose someone claims to have experienced a miracle. Antecedently, you thought it very unlikely for a miracle to occur. How confident should you be that a miracle occurred in light of this purported eyewitness testimony?

Using Bayes' Theorem, let's calculate your posterior.

M	…	The miracle occurred.
“M”	…	The eyewitness testifies that the miracle occurred.

$$c(M \mid \text{“}M\text{”}) = \frac{c(\text{“}M\text{”} \mid M) \cdot c(M)}{c(\text{“}M\text{”})}$$

Let's suppose that you think it very likely that, if a miracle did in fact occur, the eyewitness would tell you about it.

$c(\text{“}M\text{”} \mid M) = .99$

As mentioned, though, you antecedently think it is very unlikely that a miracle occurred.

$c(M) = \frac{1}{100{,}000}$

Furthermore, because there are "many instances of forged miracles," you think think there is a fairly small but non-negligible chance that the eyewitness might claim to have experienced a miracle when, in fact, no miracle has occurred.

$c(\text{“}M\text{”} \mid \neg M) = \frac{1}{1{,}000} = .001$

$$c(M \mid \text{“}M\text{”}) = \frac{c(\text{“}M\text{”} \mid M) \cdot c(M)}{c(\text{“}M\text{”} \mid M) \cdot c(M) + c(\text{“}M\text{”} \mid \neg M) \cdot c(\neg M)}$$

$c(\neg M) = 1 - c(M) = \frac{99{,}999}{100{,}000}$

$$= \frac{.99 \cdot \frac{1}{100{,}000}}{.99 \cdot \frac{1}{100{,}000} + .001 \cdot \frac{99{,}999}{100{,}000}}$$

$$= \frac{.0000099}{.0000099 + .00099999} = \frac{.0000099}{.00100989} \approx .0098$$

You should become more confident that a miracle occurred, but your credence is still fairly low—less than 1%.

So, what's the difference between the two cases?

The difference comes down to how confident you are in receiving the evidence in the respective cases.

$c(\text{“}\#n\text{”}) = \frac{1}{100{,}000} = 0.00001$

$c(\text{“}M\text{”}) = 0.00100989$

You are more confident that you'll come across testimony of miracles than you are that the newspaper would print exactly *that* number.

Base Rate Fallacy

Consider the following scenario:

> There is a particular disease D that afflicts 1 in 1,000 people. We've developed a test for the disease, which is 90% in the following sense: if one has the disease, the test comes back positive 90% of the time; if one doesn't have the disease, the test comes back negative 90% of the time.
>
> You randomly select a person and administer the test. It comes back positive. How confident should you be that they have the disease?
>
> (a) Between 90%–100%.
> (b) Between 60%–90%.
> (c) Between 30%–60%.
> (d) Between 1%–30%.
> (e) Under 1%.

In other words,

- The test's *sensitivity* (or *true positive rate*) is: $c(P \mid D) = 90\% = 0.9$,
- The test's *specificity* (or *true negative rate*) is: $c(\neg P \mid \neg D) = 90\% = 0.9$.

Try using Bayes' Theorem to figure it out!

Because $c(\neg P \mid \neg D) = .9$, we can compute the test's *false positive rate*: $c(P \mid \neg D) = 0.1$.

$$c(D \mid P) = \frac{c(P \mid D) \cdot c(D)}{c(P \mid D) \cdot c(D) + c(P \mid \neg D) \cdot c(\neg D)}$$
$$= \frac{.9 \cdot \frac{1}{1{,}000}}{.9 \cdot \frac{1}{1{,}000} + .1 \cdot \frac{999}{1{,}000}} = \frac{.9 \cdot .001}{.9 \cdot .001 + .1 \cdot .999} \approx .009 = 0.9\%$$

Even though the test is very reliable, because the base rate of the disease is so low, you should remain very confident that the person *doesn't* have it even when the test comes back positive!